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Generalized Stefan–Boltzmann Law

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Thermodynamics arguments have been employed to derive how the energy density ρ depends on the temperature T for a fluid whose pressure p obeys the equation of state $p = (\gamma - 1)\rho$, where γ is a constant. Three different methods, among them the one considered by Boltzmann (Carnot cycle), lead to the expression $\rho = \eta T^{\gamma/(\gamma-1)}$, where η is a constant. This result also appears naturally in the framework of general relativity for spacetimes with constant spatial curvature. Some particular cases are vacuum ($p = -\rho$), cosmic strings ($p = -\frac{1}{3}\rho$), radiation ($p = \frac{1}{3}\rho$), and stiff matter ($p = \rho$). It is also shown that such results can be adapted for blackbody radiation in N spatial dimensions.

1. INTRODUCTION

The Stefan–Boltzmann law has recently been discussed in the context of black-hole thermodynamics (Lavenda and Dunning-Davies, 1990). As is well known from black-hole thermodynamics, the energy density is inversely proportional to the temperature. It was then assumed that such systems behave like a perfect fluid with negative pressure $p = -\frac{1}{2}\rho$, where ρ is the energy density. Indeed, thermodynamic states with negative pressure are metastable, but they are not forbidden by general relativity nor by any other law of nature. These states are usually connected with phase transitions in the same fashion as occur for an overheated van der Waals liquid, although its overall existence has been established by different theoretical arguments (Landau and Lifschitz, 1985; Sakharov, 1982; Danielewicz, 1979; Lucács and Martinás, 1984; Lima *et al.*, 1988). For instance, as shown by Gliner (1966) and Zeldovich (1968), Lorentz invariance of the vacuum state in quantum field theories requires an energy-

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momentum tensor (EMT) of the form $T^\mu_\nu = \rho \times \mathrm{diag}(1, 1, 1, 1)$, making clear that the vacuum state can be envisaged as a particular relativistic perfect simple fluid whose equation of state is $p = -\rho$. Another example frequently considered in the literature is provided by cosmic strings (for review see Vilenkin, 1985). A randomly oriented distribution of infinitely thin strings, averaged over all directions, has an EMT of the form (Vilenkin, 1981) $T^\mu_\nu = \rho \times \mathrm{diag}(1, 1/3, 1/3, 1/3)$, so that such a system can effectively be described as a relativistic perfect fluid with $p = -\frac{1}{3}\rho$.

In cosmology it is quite usual to describe the cosmic fluid for arbitrary stages of the universe by a "gamma-law" equation of state

$$p = (\gamma - 1)\rho \tag{1}$$

where the "adiabatic index" γ lies in the interval [0, 2]. It turns out that this generalized equation of state accounts for a one-parametric family of thermodynamic systems, including those with negative pressure. In particular, the case $\gamma = 0$ is the vacuum state (Gliner, 1966; Zeldovich, 1968), whereas $\gamma = 2$ accounts for the Zeldovich stiff matter (Zeldovich, 1962). These limit cases can also be established from causality requirements, since the speed of sound is $v = c|(\partial p/\partial \rho)_\sigma|$, where c is the speed of light and σ is the specific entropy per particle. Despite such interpretations, this one-parameter family of media may alternatively be viewed as the simplest generalization of the radiation fluid ($\gamma = \frac{4}{3}, p = \frac{1}{3}\rho$), thereby leading to the question about a generalized Stefan–Boltzmann law, which would be satisfied by each one of its members. In this paper we are mainly interested in these kind of γ-fluids and, using only thermodynamic considerations, we will show by four correlated, but somewhat different approaches that $\rho(T) = \eta T^{\gamma/(\gamma-1)}$, where η is a γ-dependent constant. To this end we will repeat the same arguments historically applied to the case of blackbody radiation.

2. STEFAN–BOLTZMANN TYPE LAW FOR γ-FLUIDS

The theoretical deduction of what is nowadays known as the Stefan–Boltzmann law was given by Boltzmann in 1884. His approach was based on nothing more than an application of the Carnot cycle for which thermal radiation played the role of working substance. A more pedagogical account may be found in Richtmyer (1934). In order to generalize properly Boltzmann's result, we will replace the thermal radiation by a γ-fluid obeying equation (1). As is well known, the Carnot cycle consists of two isothermal alternated with two adiabatic reversible processes. During the isothermal process at the higher temperature T_{H} a certain quantity of heat

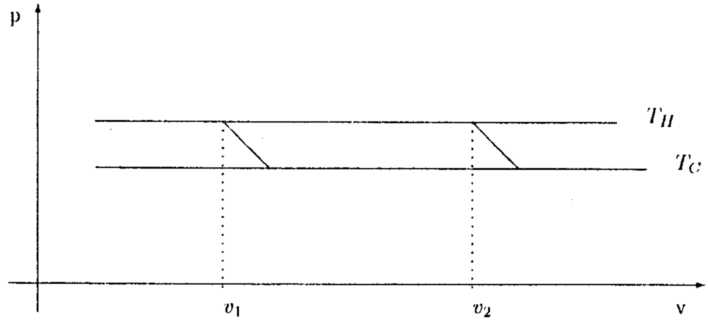

Fig. 1. The Carnot cycle for a γ-fluid.

Q_H is adsorbed by the fluid, while a quantity Q_C of heat is released along the lower isotherm T_C. The efficiency ϵ of the machine is defined as

$$\epsilon = \frac{|W|}{|Q_H|} \tag{2}$$

where $|W|$ is the net work performed by the cycle. The energy density ρ_H of the γ-fluid must remain constant in the course of the expansion from volume v_1 to the volume v_2, i.e., along the T_H isotherm. In order to counterbalance the expansion, as well as to compensate the work done by the γ-fluid itself, a quantity of heat

$$Q_H = \gamma\rho_H(v_2 - v_1) \tag{3}$$

must be supplied to the system. In the $p \times v$ diagram this process is represented by the horizontal line T_H (see Fig. 1). Next the γ-fluid undergoes an adiabatic expansion accompanied by a decrease in its energy density until reaching the value ρ_C corresponding to the lower temperature T_C. By considering that such a process connects two infinitesimally close isotherms, i.e., $dT = T_H - T_C$, it follows from equation (1) that the pressure changed by $dp = (\gamma - 1)\, d\rho$, where $d\rho = \rho_H - \rho_C$. In order to compute the work done in the infinitesimal cycle, the system is now subjected to an isothermal compression at T_C followed by an adiabatic reduction of its volume, returning to the initial value v^1. The net work performed during the cycle is given by the area of the parallelogram of Fig. 1 which, for small changes in pressure, may be approximated by $(v_2 - v_1)\, dp$. So, from the above results one can write

$$dW = (\gamma - 1)(v_2 - v_1)\, d\rho \tag{4}$$

whereas, due to equation (3), the efficiency ϵ given by (2) takes the form

$$\epsilon = \frac{(\gamma - 1)}{\gamma} \frac{d\rho}{\rho} \tag{5}$$

where the subscripts have been dropped. Now, recalling that ϵ can be computed from $\epsilon = (T_{\mathrm{H}} - T_{\mathrm{C}})/T_{\mathrm{H}}$ and considering that the cycle is infinitesimal, it follows that $\epsilon = dT/T$ and (5) can be rewritten as

$$\frac{d\rho}{\rho} = \frac{\gamma}{\gamma - 1} \frac{dT}{T} \tag{6}$$

Therefore, for $\gamma \neq 1$, a straightforward integration yields

$$\rho = \eta T^{\gamma/(\gamma - 1)} \tag{7}$$

where η is a γ-dependent constant. This generalized Stefan–Boltzmann law encompasses all the possibilities discussed earlier. In particular, in the case of black holes it is enough to take $\gamma = 1/2$.

The above result can be more easily deduced using the thermodynamic second law

$$T\,dS = dU + dW \tag{8}$$

In this case one can imagine a cylinder with ideally reflecting walls inside which there is a γ-fluid at a temperature T. The volume V can be reversibly changed by moving a piston. From the "γ-law" equation of state, the work dW done by the system during the infinitesimal variation dV is $dW = (\gamma - 1)\rho\, dV$. Hence, by considering that $U = \rho(T)V$, one obtains from (8)

$$dS = \frac{V}{T}\left(\frac{d\rho}{dT}\right) dT + \frac{\gamma\rho}{T}\, dV \tag{9}$$

and, since dS is an exact differential, it follows from (9) that

$$\frac{d\rho}{\rho} = \frac{\gamma}{\gamma - 1} \frac{dT}{T} \tag{10}$$

which, after a straightforward integration, yields again equation (7).

Perhaps the more direct proof of the generalized Stefan–Boltzmann law is obtained from Gibbs–Duhem relation in the entropic representation. For a one-component thermodynamic system it is given by (Callen, 1985)

$$U\,d\left(\frac{1}{T}\right) + V\,d\left(\frac{p}{T}\right) - N\,d\left(\frac{\mu}{T}\right) = 0 \tag{11}$$

where μ is the chemical potential and N is the number of particles. As in the case of radiation ($\gamma = 4/3$), we assume that any γ-fluid has $\mu = 0$, so that the above relation takes the form

$$\frac{dp}{\rho + p} = \frac{dT}{T} \tag{12}$$

and from equation (1) one has again

$$\frac{d\rho}{\rho} = \frac{\gamma}{\gamma - 1}\frac{dT}{T} \tag{13}$$

It should be remarked that, unlike the earlier deductions, in the above method it was not necessary to make the assumption $\rho = \rho(T)$, a fact already noticed in Lavenda and Dunning-Davies (1990).

Now, for the sake of completeness, we will show how the generalized Stefan–Boltzmann law can be inferred using the Einstein field equations (EFE) applied to the cosmological domain. The line element of a homogeneous and isotropic self-gravitating fluid in general relativity is (Weinberg, 1972)

$$ds^2 = c^2\,dt^2 - R^2(t)\left(\frac{dr^2}{1 - \kappa r^2} + r^2\,d\theta^2 + r^2 \sin^2\theta\,d\phi^2\right) \tag{14}$$

where $\kappa = 0, \pm 1$ is the curvature parameter and $R(t)$ is the scale function. For a perfect fluid the energy conservation law in such a background takes the form

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} = 0 \tag{15}$$

where a dot means time derivative. Now, using the "γ-law" equation of state, the above equation is immediately integrated as

$$\rho = \rho_0\left(\frac{R_0}{R}\right)^{3\gamma} \tag{16}$$

where ρ_0 and R_0 are constants. On the other hand, in the same conditions, the temperature evolution equation reads (Weinberg, 1971; Calvão *et al.*, 1992)

$$\frac{\dot{T}}{T} = -3\left(\frac{\partial p}{\partial \rho}\right)_n \frac{\dot{R}}{R} \tag{17}$$

and from equation (1) one obtains

$$T = T_0\left(\frac{R_0}{R}\right)^{3(\gamma - 1)} \tag{18}$$

Therefore, combining equations (16) and (18), it follows that

$$\rho = \eta T^{\gamma/(\gamma - 1)} \tag{19}$$

in accordance with our previous results. It should be noticed that the latter deduction is not independent of the second one insofar as equation (17) has been established using the fact that $d\sigma$ is an exact differential (σ is the specific entropy). Although the approach presented here is not new, equation (19) is apparently not well known. In particular, this explains why the generalized Stefan–Boltzmann law was not inferred from cosmology. In general only the case of radiation ($\gamma = 4/3$) is used to describe the early stages of the universe.

It is also interesting to examine how the temperature of a γ-fluid depends on the volume during an adiabatic expansion. Indeed, such relation was earlier derived in Lima and Maia (n.d.), but now it can be more easily obtained using the generalized Stefan–Boltzmann law. In fact, by replacing (7) into (9) and using that $dS = 0$, we arrive at

$$\frac{1}{\gamma - 1} \frac{dT}{T} + \frac{dV}{V} = 0 \tag{20}$$

and therefore

$$T^{1/(\gamma - 1)} V = \text{const} \tag{21}$$

The remarkable difference in the thermal behavior presented by the two subclasses of γ-fluids, which are naturally separated by the singular case $\gamma = 1$, will appear most clearly upon comparing the expressions (19) and (21). According to equation (21), for $\gamma > 1$, the γ-fluid cools in the course of the expansion and from (19), its energy density will diminish proportionally to a positive power of the temperature. However, if $\gamma < 1$, the temperature grows, whereas the energy density now scales with a negative power of the temperature, thereby decreasing again for a decreasing of ρ if $\gamma \neq 0$. For instance, in the case of cosmic strings ($\gamma = 2/3, p = -\frac{1}{3}\rho$), one has $T \propto V^{-1/3}$ and $\rho \propto T^{-2}$. The limit case $\gamma = 0$ is the vacuum state ($p = -\rho$), for which ρ remains constant and $T \propto V$.

To emphasize the interest and generality of our results some points deserve comment. First, we remark that if one tries to apply the second law for each monochromatic component of the generalized radiation in the form

$$T\, dS_\nu = d(\rho_\nu V) + p_\nu\, dV \tag{22}$$

the related Euler expression

$$S_\nu = \frac{1}{T}(\rho_\nu + p_\nu) V \tag{23}$$

will be valid only if $p_\nu(T)$ and $\rho_\nu(T)$ satisfy the "Gibbs–Duhem" equation

$$d\left(\frac{p_\nu}{T}\right) + \rho_\nu\, d\left(\frac{1}{T}\right) = 0 \tag{24}$$

whose integral is

$$p_\nu(T) = T \int_0^T \frac{\rho_\nu(T)}{T^2}\, \mathrm{d}T \tag{25}$$

In fact, by integrating the above expression over all frequencies, the "γ-law" equation of state is recovered. This generalizes the standard result for blackbody radiation (Lavenda and Dunning-Davies, 1990). Second, in higher-dimensional theories, such as those of Kaluza–Klein type (Appelquist *et al.*, 1987), the equation of state of the radiation in N spatial dimensions is taken to be $p = \rho/N$. In this case, it is easy to see that the generalized Stefan–Boltzmann law given by (7) and the corresponding deductions presented here can be adapted up to some slight identifications and/or modifications. First of all, it is necessary to fix $\gamma > 1$, since the number of spatial dimensions is positive definite. Now, by taking $\gamma = 1 + 1/N$, one obtains from equation (7) that

$$\rho = \eta T^{N+1} \tag{26}$$

which describes the Stefan–Boltzmann law for N spatial dimensions. Of course, in order to deduce such an equation using general relativity, one must consider the EFE in $N + 1$ dimensions. Finally, we remark that the existence of the generalized Stefan–Boltzmann law points to the possibility of an enlargement of the standard Wien law in order to encompass the class of γ-fluids, now regarded as generalized radiation. In this regard, we notice that an attempt was recently made to fix a Wien-type law to the vacuum state (Lima and Maia, n.d.). It is easy to see that the spectrum proposed, namely $\rho_T(\nu) = \nu^{-1} f(T/\nu)$, where f is an arbitrary function of its arguments, is compatible with our equation (7). Further, since in the case of photons ρ scales with ν^3, it should be expected that for arbitrary values of γ, $\rho_T(\nu)$ scales with $\nu^{1/(\gamma-1)} f(T/\nu)$. This problem will be discussed in detail elsewhere.

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